These are the problems and solutions (except for 3) of Panama's National Olympiad Final Round Level B.

These are my solutions as I wrote them on the test (except for 3).

Problem 1.

Let a and b be positive integers such that $a^3 - b^3 = 485$.Find $a^3 + b^3$.

Solution

From $a^3 - b^3 = 485$ we get:

$(a - b)(a^2 + ab + b^2) = 5 * 97$

Hence

$a - b = 5$

$a^2 + ab + b^2 = 97$

Now since $a^2 + ab + b^2 - (a - b)^2 = 3ab = 97 - 25 = 72$ it follows that $ab = 24$. Hence we get the system of equations:

$a - b = 5$

$ab = 24$

With solution $a = 8, b = 3$. Therefore, $a^3 + b^3 = 539$.

Problem 2. Consider two lines perpendicular to each other such that the unit circle is tangent to them.Now draw a circle that is tangent to the unit circle and both lines.Now draw another circle such that it is tangent to the previously drawn circle and both lines.Find the radius of the tenth circle drawn in this process.

Solution

Consider the common tangent of the unit circle and the next circle.

Let A be the point of intersection of the perpendicular lines.

And let B and C be the points of intersection of the perpendicular lines with the common tangent to the unit circle and the next circle.

Since this process is iterative it suffices to find the radius of the circle drawn after the unit circle.

Consider triangle ABC , the unit circle is just the excircle of ABC and the next circle is just the incircle of ABC.

Since triangle ABC is isosceles with $AB = AC$ let M be the midpoint of BC. Now consider the excenter I_a.

Since $AI_a = \sqrt{2}$,and A, M and I_a are collinear it follows that $AM = \sqrt{2} - 1$.

Now consider triangle ABM , it is isosceles with $AM = BM$. Hence by the Pythagorean theorem $AB = 2 - \sqrt{2}$.

Therefore the area of ABC , $[ABC] = 3 - 2\sqrt{2}$. Also, it is well known that $sr = [ABC]$ where s is the semiperimeter and r is the inradius.

Now $s = \frac{2(2-\sqrt{2}+\sqrt{2}-1)}{2} = 1$.

Hence $r = 3 - 2\sqrt{2}$. So it follows that the radius of the tenth circle is just $(3 - 2\sqrt{2})^9$.

Problem 3.

Consider a $(2k + 1) * (2k + 1) * (2k + 1)$ cube whose surface is painted grey and black such that the corners are grey.

i) Find in terms of k the amount of $1 * 1 * 1$ cubes such that exactly two of their faces is grey. ii) Find k such that the amount of $1 * 1 * 1$ cubes with at least 2 faces grey is 2012.

Solution

I wrote an explanation for this on the test but it is trivial so I'll just put the answer:

i)This is just $12k - 12$.

ii)$k = 168$.