1 Introduction

This is a set of problems from various mathematics competitions I used to train students for Panama's National Olympiad. The given problems are much harder than those that usually arise in Panama's National Olympiad; however I liked the set of problems so I used it and I hope they learn a lot of mathematics through this problems. Good luck and happy problem solving!!

2 Algebra

Problem 1. Given a subset A of the set $\{1, 2, 3, ..., n\}$, let sum of A be the sum of all the elements in A. We wish to partition the set of integers $\{1, 2, 3, ..., n\}$ in twelve distinct and non-empty subsets that such that they all have the same sum .Determine the least n for which this is possible.

Problem 2. Let $A = \{1, 2, 3, ..., 2^n\}$. Consider the biggest odd divisors of each of the elements of A and take their sum. Find a closed form expression for this sum.

Problem 3.Show that every positive integer can be written as $a^2 + b^2 - c^2$ for a, b, c nonnegative integers with $a < b < c$.

Problem 4.Determine every positive integer n such that there exists $k \geq 2$ positive rational integers $a_1, a_2, a_3, ..., a_k$ such that :

$a_1 + a_2 + ... + a_k = a_1 * a_2 * ... * a_k = n.$

3 Number Theory

Problem 1.Let a, b, c, d be positive integers such that $ab = cd$. Show that $a + b + c + d$ is not prime.

Problem 2.Determine every positive integer (m, n) that satisfies the following equation:

$\frac{1}{m} + \frac{1}{n} - \frac{1}{mn^2} = \frac{3}{4}.$

Problem 3.Find a list of 5 distinct prime numbers where the difference between any two consecutive terms of the list is 6. Show that this list is unique.

Problem 4. Show that if a positive integer n is a power of 2, then every set formed with $2n - 1$ positive integers it is possible to choose n of them such that their sum is divisible by n.

Problem 5. Consider the numbers of the form 101010...101 that start and finish with the digit 1, and that are formed by alternating the ones and the zeroes . Determine all integers of this form that are also prime numbers.

Problem 6. Show that every integer can be written as the sum of five perfect cubes.

4 Geometry

Problem 1.Let C be the point of tangency of circumference ω and line l, and let AB be a diameter of ω (A and B are different from C). Let N be the foot of the perpendicular from C to AB. Through a point F in segment CN (F is different from C and N), it is drawn a parallel line to CB that intersects l at E and CA at G. Show that $EG = GF$.

Problem 2. Let P and Q be points on the side AB of triangle ABC (P between A and Q), such that $\angle ACP = \angle PCQ = \angle QCB$.Let M and N be the points of intersection of lines CP and CQ with the bisector AD of $\angle BAC$,respectively. If $NP = CD$ and $3\angle A = 2\angle C$, show that $[CDQ] = [QNB]$.

Problem 3. A cuadrilateral $ABCD$ is such that the midpoint of BC is the center of the semicircle that is tangent to the sides AB, CD and DA.Find the length of BC in terms of AB and CD.

Problem 4. Let Ω and Γ be two circumferences such that the center O of Ω lies on Γ. Let C and D be the points of intersection of the circumferences.Take a point A on Ω and a point B on Γ such that AC is tangent to Γ at C and BC is tangent to Ω at C.AB intersects Γ at E and Ω at F. CE intersects Ω at G and CF intersects GD at H. Prove that the point at which GO and EH intersect is the circumcenter of triangle DEF.

5 Combinatorics

Problem 1.In a $10 * 10$ grid an integer number is written in each box in such a way that the difference between adjacent boxes is less than or equal to 1.

(a) Find a grid that has the biggest possible amount of distinct numbers and explain why there can't be a board that has more distinct numbers.

(b) Show that in the grid there is number that repeats itself more than 6 times.

Problem 2. Five children divide into groups and in each group they take hands forming a dancing spinning wheel.

How many different tires can the children form, if it is valid to have groups of 1 through 5 kids, and there may be any number of groups?

Problem 3.Consider 50 points in the plane such that no three points are collinear. Each of these points is colored by one of 4 colors.

Show that the number of escalene triangles of the same color is at least 130.

Problem 4. Let $n \geq 2$ be an integer . In how many ways is it possible to arrange every number from $\{1, 2, 3, ..., 2n\}$ in the boxes of a $2 * n$ grid , one in each box , in such a way that any two consecutive numbers are in adjacent boxes of the grid?

Problem 5. For which integers $n \geq 5$ is possible to paint the vertices of a regular n-gon using at most 6 colors in such a way that any 5 consecutive vertices have distinct colors.

Problem 6. In a $5 * 5$ grid's boxes there is written either a 1 or -1. Then the product of the numbers in each row and column is calculated. Show that the sum of 10 of those products is not 0.